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Technical Note

Multilayer differential discrete ordinate method for inhomogeneous participating media

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Abstract

This paper presents a multilayer differential discrete ordinate method to solve the radiative transfer equation for an absorbing, emitting and scattering inhomogeneous plane parallel medium. This method reduces the integro-differential equation into a set of coupled first order ordinary differential equations with two point boundary conditions on using a suitable quadrature scheme. These equations are then solved numerically. Numerical validation of the method for gray medium is done by comparing the results obtained with benchmark cases available in the literature. Validation for a non-gray medium is done by considering a problem concerning radiative transfer from the atmosphere. The brightness temperature at the top of the atmosphere is calculated at various frequencies and validated with those obtained by several other numerical methods.

Keywords: Multilayer plane parallel media; Differential discrete ordinates method; Atmospheric radiative transfer

1. Introduction

The propagation of radiation through a participating medium has numerous engineering applications such as atmospheric remote sensing, industrial furnaces, high temperature porous materials, rocket exhaust plumes, fluidized bed combustors and ablation systems on reentry vehicles. It is the presence of a medium that complicates the analysis of radiative transfer. The difficulty arises in handling the three-dimensional nature of radiation combined with complex mechanisms of absorption, emission and scattering.

The governing equation for radiative transfer in a participating medium is an integro-differential equation in terms of intensity. The solution of such an equation by exact techniques is extremely difficult except for some idealized situations related to the nature of the medium and geometry of the problem. Hence, extensive research has been done in the development of numerical methods to solve the radiative transfer equations for more involved cases.

Exact solution techniques for the integral equation of transfer are available for simple cases [1]. In the recent past, various numerical techniques have been developed which can easily incorporate important effects such as anisotropy and non-uniform properties, by maintaining accuracy and improving computational efficiency.

Several techniques for solving the radiative transfer equation are currently available, as for example Monte Carlo method [2], F_N method [3], spherical harmonics method [4], two-flux methods [5], finite volume method [6], discrete ordinates methods [7–9]. These techniques employ complicated mathematics and they generally require extensive programming effort.

The discrete ordinates method solves the radiative transfer equation by reducing it into a set of differential equations. Chandrasekhar [7] proposed the discrete ordinates method in his work on stellar and atmospheric radiation. The numerical analysis of the method for multiple scattering inhomogeneous media is discussed in detail by Stamnes

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Nomenclature

a_n	expansion coefficients
A	elements of matrix **A**
B	elements of the vector **B**
I	intensity, W/m^2
L	thickness of plane parallel slab, m
M	half the order of quadrature
N	degree of scattering anisotropy
P_n	Legendre polynomials
q	radiative heat flux rate, W/m^2
R	number of layers
T	temperature, K
w	weights in the numerical quadrature
z	Cartesian coordinate, m

Greek symbols

δ_{ij}	Kronecker's delta
ε	surface emissivity
θ	polar angle, radians
μ	direction cosine with respect to z direction, $\cos\theta$
ρ	reflectivity
σ	radiative coefficient
τ	optical depth, m^{-1}
υ	frequency of incident radiation, GHz
ϕ	azimuthal angle, rad
Φ	scattering phase function
ω	single scattering albedo $= \sigma_s/(\sigma_a + \sigma_s)$

Subscripts

a	absorption
B	blackbody
inc	incident radiation
L	depth of medium
r	layer number
s	scattering
0	coordinate origin

Superscripts

+	upward or positive direction
−	downward or negative direction

et al. [8] and Chalhoub [9]. Fiveland [10] demonstrated various quadrature schemes for the discrete ordinates method and verified the stability of solution under each scheme. The accuracy of the method is greatly affected by the quadrature employed for angular discretization.

Kumar et al. [11] proposed the differential discrete ordinate method (DDOM) for homogeneous media to solve the one-dimensional radiative transfer equation. The methodology involves the use of a quadrature scheme of the discrete ordinates method to reduce the radiative transfer equation into a set of ordinary differential equations. The resultant system of equations is solved by readily available software routines. This reduces the need for complicated mathematics and tedious programming effort required on the part of the user. Also, variations in the boundary conditions and energy equilibrium conditions of the medium can be easily incorporated. The results of this method for various quadrature schemes are reported and they agree with benchmark cases available in the literature, when a suitable quadrature scheme is used. This method can be easily implemented with least programming effort without compromising on the accuracy.

In many practical cases like radiative transfer from the atmosphere, heat transfer in furnaces and so on, the intervening medium is inhomogeneous i.e., the single scattering albedo and phase function vary with the optical depth of the medium. The problem of inhomogeneous medium is dealt with by dividing the medium into a number of homogeneous layers such that the single scattering albedo and phase function in each layer are constant. The objective of this paper is to implement the differential discrete ordinate method to an inhomogeneous medium like the atmosphere.

The accuracy of the method for inhomogeneous media is tested by considering three different cases for which exact or benchmark results are available. The method is also applied to an atmosphere radiative transfer problem, where the medium is non-gray and the results are compared with those obtained by several other numerical methods.

2. Formulation

The equation describing the transfer of monochromatic radiation at a frequency υ through a plane parallel medium (Fig. 1) is given by

$$\mu\frac{\mathrm{d}I_\upsilon(\tau_\upsilon,\mu,\phi)}{\mathrm{d}\tau_\upsilon} = -I_\upsilon(\tau_\upsilon,\mu,\phi) + (1-\omega_\upsilon)I_{\mathrm{B}\upsilon}[T(\tau_\upsilon)] + \frac{\omega_\upsilon}{4\pi}\int_{\mu'=-1}^{+1}\int_{\phi'=0}^{2\pi} I_\upsilon(\tau_\upsilon,\mu',\phi')\Phi_\upsilon \times (\mu',\phi';\mu,\phi)\,\mathrm{d}\mu'\,\mathrm{d}\phi' \quad (1)$$

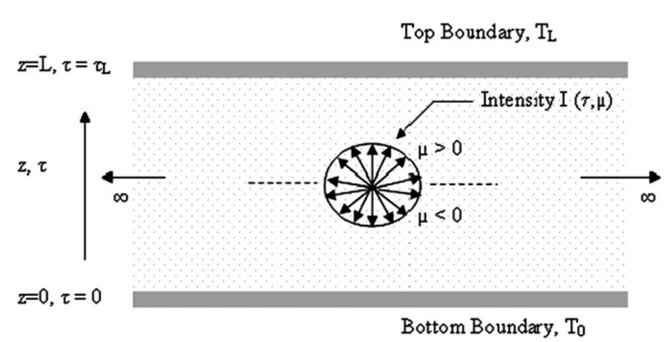

Fig. 1. Schematic of a plane parallel medium.

Here $I_\upsilon(\tau_\upsilon, \mu, \phi)$ is the monochromatic intensity in the direction μ, ϕ at optical depth τ_υ, μ is the cosine of the polar angle θ, ϕ is the azimuthal angle, $I_{B\upsilon}$ is the Planck function at a given frequency υ and temperature T, ω_υ is the single scattering albedo and $\Phi_\upsilon(\mu', \phi'; \mu, \phi)$ is the scattering phase function. Eq. (1) is also valid in terms of total quantities for a gray medium when integrated throughout the frequency range. In such a case, the subscript υ can be dropped off from Eq. (1).

The subscript υ is omitted hereafter to simplify the notation and it is invoked whenever required. The azimuthally independent case of Eq. (1) is considered for further analysis and the corresponding equation for this case is given by

$$\mu \frac{\mathrm{d}I(\tau,\mu)}{\mathrm{d}\tau} = -I(\tau,\mu) + (1-\omega)I_{\mathrm{B}} + \frac{\omega}{2}\int_{\mu'=-1}^{+1} I(\tau,\mu\prime).\Phi(\mu';\mu).\mathrm{d}\mu' \tag{2}$$

The scattering phase function for the case of anisotropic scattering can be approximated as a truncated Legendre series as,

$$\Phi = \sum_{n=0}^{N} a_n P_n(\mu) P_n(\mu'), \quad a_0 = 1 \tag{3}$$

where P_ns are the Legendre polynomials, a_ns are expansion coefficients of the phase function and N is the degree of anisotropy of scattering.

2.1. Multilayer medium

The single scattering albedo ω (τ) and phase function $\Phi(\mu';\mu)$ depend on the spatial location of the medium when the medium is inhomogeneous. For the case of an inhomogeneous medium, the medium is divided into R homogeneous layers in which the single scattering albedo and the phase function are taken to be constant. A schematic of the multi layer model is shown in Fig. 2. The equation of transfer for any homogeneous layer r within the boundaries τ_{r-1} and τ_r can be written from Eq. (2) as,

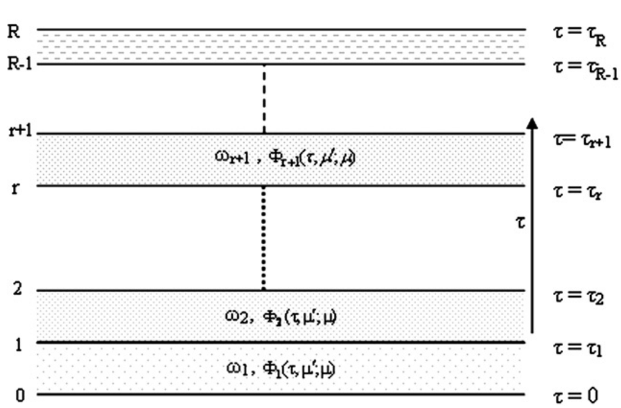

Fig. 2. Schematic of a multilayer medium.

$$\mu \frac{\mathrm{d}I_r(\tau,\mu)}{\mathrm{d}\tau} = -I_r(\tau,\mu) + (1-\omega)I_{\mathrm{B},r} + \frac{\omega_r}{2}\int_{\mu'=-1}^{+1} I_r(\tau,\mu\prime).\Phi_r(\mu';\mu).\mathrm{d}\mu' \tag{4}$$

The azimuthally independent radiative transfer equation for any homogeneous layer is analyzed by replacing the integral over μ in Eq. (4) by a quadrature with points lying between -1 and $+1$ which results in the following system of ordinary differential equations:

$$\mu_i \frac{\mathrm{d}I_{i,r}(\tau)}{\mathrm{d}\tau} = -I_{i,r}(\tau) + (1-\omega_r)I_{\mathrm{B},r}[T(\tau)] + \frac{\omega_r}{2}\sum_{\substack{j=-M\\ j\neq 0}}^{M} w_j I_{j,r}(\tau)\Phi_r(\mu_j;\mu_i), \quad i = -M,\ldots,M, \; i\neq 0 \tag{5}$$

where μ_i's are the quadrature points and w_i's are the corresponding weights of the order 2M quadrature having an even number of points. The above equations constitute a system of 2M coupled ordinary differential equations. For any layer in Fig. 2, Eq. (5) can be written in matrix form as

$$\frac{\mathrm{d}\{I\}}{\mathrm{d}\tau} = [A]\{I\} + \{B\} \tag{6}$$

where

$$A_{ij} = \frac{1}{\mu_i}\left[-\delta_{ij} + \frac{\omega_r}{2}w_j\Phi(\mu_i;\mu_j)\right],$$

$$B_i = \frac{1}{\mu_i}(1-\omega_r)I_{\mathrm{B}}[T(\tau)], \;\; i = -M \text{ to } M$$

2.2. Boundary conditions

The present method can accommodate various boundary conditions. Different boundary conditions such as diffusely emitting and reflecting boundaries, diffusely emitting and specular reflecting boundaries have been employed and the results for the specific case of diffusely emitting and reflecting boundaries are reported. The azimuthally independent boundary conditions for the multilayer problem at lower and upper boundaries are respectively expressed as

$$I_1(0,\mu) = \varepsilon_0 I_{\mathrm{B},1}(0) + 2\rho_0\int_{\mu'=0}^{1} I_1(0,-\mu).\mu'\,\mathrm{d}\mu', \quad \mu>0 \tag{7}$$

$$I_R(\tau_R,-\mu) = \varepsilon_{\tau_R} I_{\mathrm{B},R}(\tau_R) + 2\rho_{\tau_R}\int_{\mu'=0}^{1} I_R(\tau_R,+\mu).\mu'\,\mathrm{d}\mu', \quad \mu>0 \tag{8}$$

where ε and ρ are the diffuse emissivity and reflectivity of the surfaces at $\tau = 0$ and $\tau = \tau_R$, respectively. The black body intensities at the top and the bottom surfaces are given by the Planck's black body function on a spectral basis and by the Stefan–Boltzmann law on total basis.

These boundary conditions in (7) and (8) can be rewritten in the discretized form as

$$I_{+i,1}(0) = \varepsilon_0 I_{B,1}(0) + 2\rho_0 \sum_{j=-M}^{-1} w_j I_{j,1}(0)\mu_j, \quad i = 1,\ldots,M \quad (9)$$

$$I_{-i,R}(\tau_R) = \varepsilon_{\tau_R} I_{B,R}(\tau_R) + 2\rho_{\tau_R} \sum_{j=1}^{M} w_j I_{j,R}(0)\mu_j, \quad i = 1,\ldots,M \quad (10)$$

These boundary conditions indicate that half of the 2M boundary conditions required to solve the discretized radiative transfer equation are at the lower boundary and the remaining half are at the upper boundary. Thus, this is a two point boundary condition problem. As the medium is divided into a number of layers, one more set of conditions are required for the closure of the problem. This can be achieved by considering the intensities across the layer interfaces to be continuous. This continuity condition can be mathematically represented as

$$I_r(\tau_r, \pm\mu) = I_{r+1}(\tau_r, \pm\mu) \quad (10a)$$

The interface conditions actually represent matching of the intensities at the top of a lower layer with the intensities at the bottom of the upper layer. There will be M such boundary conditions that close the problem mathematically. For the satisfaction of the interface conditions, iterations go on within each layer. Typically, for an optical depth of 20 m^{-1} and 10 layers, approximately 200 iterations were required for a typical case. Upon convergence, the change in dimensionless intensities in any layer should be less than 10^{-6}.

2.3. Quadrature scheme

The present methodology is capable of accommodating different types of quadrature schemes, such as Gaussian, Lobatto, Double Gauss and Fiveland. These schemes differ in the values of μ_j and w_j. The Fiveland scheme fixes the weights and evaluates the points and hence reduces the degree of the polynomial to be evaluated to determine the points. The results for the double Gauss scheme are found to be in good agreement with the exact analytical solutions and benchmark cases when applied to the problem of a homogeneous medium. In this scheme, the evaluations of points and weights are done separately over the half-ranges $-1 < \mu < 0$ and $0 < \mu < 1$. The main advantage of this scheme is that the quadrature points are clustered both toward $|\mu| = 1$ and $\mu = 0$. The intensity varies rapidly around $\mu = 0$ and hence the clustering toward $\mu = 0$ will give accurate results [8].

2.4. Solution method

The system of coupled ordinary differential equations along with appropriate two point boundary conditions and the energy equilibrium conditions can be solved by using a commercially available DBVPFD [IMSL-FORTRAN 90] subroutine. This subroutine has already been extensively tested for numerical accuracy and stability. The subroutine employs the variable step finite-difference method. An initial approximation to the solution is required to obtain the solution. This approximate solution is corrected by a finite-difference technique with deferred correction allied with a Newton iteration to solve the finite-difference equations [12]. In a multilayer medium, initially the intensity field is assumed in each layer and then each individual layer is solved for new intensity distribution. The final intensity field is then obtained iteratively by matching the continuity condition [Eq. (10)] at each interface. An algorithm in FORTRAN is developed using DBVPFD subroutine to obtain the results using the present method.

3. Validation

The accuracy of the present method for an inhomogeneous medium is first tested for gray participating media by considering three different problems. The problems considered for comparison are: (a) two-layer model with isotropic scattering, (b) six layer model with anisotropic scattering and (c) a semi infinite isotropically scattering atmosphere. Stamnes and Conklin [13] considered these three cases to verify the numerical models proposed by them for inhomogeneous media. The applicability of the present method for atmospheric radiative transfer problems where the medium is non-gray is checked for a simplified three layer atmosphere model. Brightness temperatures (Brightness temperature is defined as the temperature that a blackbody would need to have in order to emit radiation of the observed intensity at a given wavelength) at various microwave frequencies are reported along with those available in literature.

3.1. Two layer model with isotropic scattering

The two layer model (Fig. 3) with different values of scattering albedo in each isotropic scattering layer was solved by Ozisik and Shouman [14] and the results of hemispherical reflectivity and transmissivity of the slab for externally incident isotropic radiation were reported. The hemispherical reflectivity and transmissivity are calculated from Eqs. (11) and (12) for the following conditions of the medium:

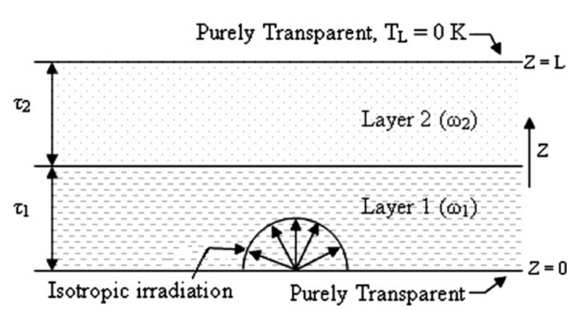

Fig. 3. Two layer model with isotropic incidence.

(1) No radiation sources in the two layers (no emission from the medium).
(2) Both the boundaries are transparent.
(3) No emission of radiation from the top boundary ($z = L$).
(4) Isotropic radiation of unit intensity is incident on the bottom transparent boundary ($z = 0$).

The hemispherical reflectivity of the slab is defined as

$$\text{Reflectivity} = \frac{q^-(0)}{q_{\text{inc}}} \tag{11}$$

The hemispherical transmissivity of the slab is defined as

$$\text{Transmissivity} = \frac{q^+(L)}{q_{\text{inc}}} \tag{12}$$

where

$$q^-(0) = 2\pi \int_{\mu=-1}^{0} I(0,\mu)\mu\,d\mu \tag{13}$$

$$q^+(L) = 2\pi \int_{\mu=0}^{1} I(\tau_L,\mu)\mu\,d\mu \quad \text{and} \tag{14}$$

$$q_{\text{inc}} = 2\pi \int_{\mu=0}^{1} 1\cdot\mu\cdot d\mu = \pi \tag{15}$$

The results from the present work for the 8 stream double Gauss quadrature are given in Table 1 for a variety of scattering albedos and slab optical depths. The results from the F_N method [14] are also included in the Table. The results obtained with DDOM agree with the results of the F_N method to the third or fourth decimal place.

3.2. Six layer model with isotropic and anisotropic scattering

Devaux et al. [15] solved a multilayer model for inhomogeneous atmospheres using the F_N method. In this model, the atmosphere is divided into six homogeneous layers with each layer having a different single scattering albedo. An anisotropic scattering cold medium is considered and the same scattering law is applied to all the layers. Both the boundaries are transparent. There is no emission from the top most boundary. The incident radiation from the external source on the bottom boundary is of the form

$$I(0,\mu) = \mu^p, \quad \mu > 0 \tag{16}$$

The single scattering albedo and optical thickness of each layer are listed in Table 2. Devaux et al. tabulated the results of the albedo and transmission factor for isotropic scattering and anisotropic scattering. The expansion coefficients of the anisotropic scattering phase function from [15] are given in Table 3. The albedo and transmission factor are expressed as

$$\text{Albedo} = \frac{q^-(0)}{q^+(0)} \tag{17}$$

where $q^-(0) = 2\pi \int_{\mu=0}^{1} I(0,-\mu)\mu\,d\mu$ and $q^+(0) = 2\pi \int_{\mu=0}^{1} I(0,\mu)\mu\,d\mu$.

$$\text{Transmission factor} = \frac{q^+(L)}{q^+(0)} \tag{18}$$

where $q^+(L) = 2\pi \int_{\mu=0}^{1} I(\tau_L,\mu)\mu\,d\mu$.

The albedo results obtained for $p = 0, 1, 2$ [Eq. (16)] using the present method for two different types of scattering are reported in Tables 4 and 5, respectively for the isotropic scattering and anisotropic scattering. Double Gauss quadrature is employed in this case.

The results obtained by the present method agree to the second or third decimal place in the case of 4-stream DDOM with the F_N method (benchmark case). The results of the 8-stream and 16-stream closely agree with those calculated by the F_N method [15].

Table 1
Reflectivity and transmissivity for a two layer model

ω_1	ω_2	τ_1	τ_2	Reflectivity		Transmissivity	
				DDOM[a] (Present)	F_N method [14]	DDOM[a] (Present)	F_N method [14]
0.8	0.95	0.25	0.25	0.2251	0.2252	0.6504	0.6503
0.6	0.5	0.25	0.25	0.1278	0.1278	0.5476	0.5474
0.5	0.3	0.25	0.25	0.0930	0.0930	0.5131	0.5128
0.8	0.95	0.5	0.5	0.3057	0.3056	0.4597	0.4597
0.6	0.5	0.5	0.5	0.1661	0.1661	0.3205	0.3206
0.5	0.3	0.5	0.5	0.1219	0.1219	0.2834	0.2835
0.8	0.95	1.0	1.0	0.3509	0.3509	0.2476	0.2476
0.6	0.5	1.0	1.0	0.1877	0.1877	0.1164	0.1164
0.5	0.3	1.0	1.0	0.1398	0.1398	0.0930	0.0930
0.8	0.95	1.0	2.0	0.3786	0.3786	0.1600	0.1600
0.6	0.5	1.0	2.0	0.1892	0.1892	0.0420	0.0419
0.5	0.3	1.0	2.0	0.1402	0.1402	0.0300	0.0301

[a] 8 stream.

Table 2
Single scattering albedo and optical depth for each layer

Layer, r	ω_r	τ_r
1	0.65	1
2	0.70	2
3	0.75	3
4	0.80	4
5	0.85	5
6	0.90	6

Table 3
Coefficients (a_n) of phase function

n	a_n
0	1
1	2.00916
2	1.56339
3	0.67407
4	0.22215
5	0.04725
6	0.00671
7	0.00068
8	0.00005

Table 4
Albedo for six layer model for isotropic scattering

$p^{\dagger}$	DDOM (Present)			F_N method [15]
	4-stream	8-stream	16-stream	
0	0.2294	0.2280	0.2280	0.2280
1	0.2141	0.2148	0.2148	0.2148
2	0.2090	0.2079	0.2079	0.2079

† Refer Eq. (16).

Table 5
Albedo for six layer model for anisotropic scattering

$p^{\dagger}$	DDOM (Present)			F_N method [15]
	4-stream	8-stream	16-stream	
0	0.10290	0.10007	0.10005	0.10010
1	0.07937	0.08057	0.08058	0.08059
2	0.07152	0.07052	0.07053	0.07051

† Refer Eq. (16).

3.3. Isotropically scattering semi infinite atmospheres

Garcia and Siewert [16] used the F_N method to solve the problem of isotropically scattering semi infinite atmosphere with an exponentially varying single scattering albedo and isotropic incidence at one boundary. The single scattering albedo is expressed as,

$$\omega = \omega_0 \exp(-\tau/S) \quad 0 \leqslant \omega_0 \leqslant 1 \quad \text{and} \quad S > 0 \tag{19}$$

This problem was solved by Stamnes and Conklin [13] using the discrete ordinates approach by dividing the atmosphere into a suitable number of homogeneous layers. The expressions for scattering albedo for each layer ω_r and layer optical thickness $\Delta\tau_r$ are given as [13]

$$(\Delta\tau)_r = \tau^* \frac{\exp(r/KN)}{KN\exp(1/N-1)} \tag{20}$$

$$\omega_r = \omega_0 \exp(-(\tau_p + \tau_{p-1})/2S), \quad p = 1, 2 \ldots, K \tag{21}$$

where K is a shape factor which determines the layer thickness distribution, N is the total number of layers and τ^* is the total optical thickness of the atmosphere. Calculations are done for $K = 0.2$, $\tau^* = 30$ and for $S = 1, 10, 100$. Smaller values of S indicate the most rapid variation of scattering albedo with optical depth. The present multilayer method adopts these expressions and the albedos [Eq. (17)] for 10, 50 and 100 homogeneous layers are calculated by using the double Gauss quadrature scheme for angular discretization. The albedo results are given in Table 6 for two values of ω_0 and for the 8-stream and 16-stream quadrature. The results obtained from the F_N method [16] are also given in Table 6.

The maximum error in albedo associated with 10 homogeneous layers is about 4%. The results of 8-stream and 16-stream quadratures for 100 homogeneous layers agree to the fourth decimal place of results of Garcia and Siewert. This shows that finer refinement of the homogeneous layers to approximate the inhomogeneous medium promises better accuracy.

The three different cases that test the accuracy of the multilayer model show that this model gives fairly accurate solutions when compared with the benchmark solutions (F_N method). The time taken to obtain the solution is less when more number of layers is used. This method gives good results for the case of anisotropically scattering inhomogeneous media.

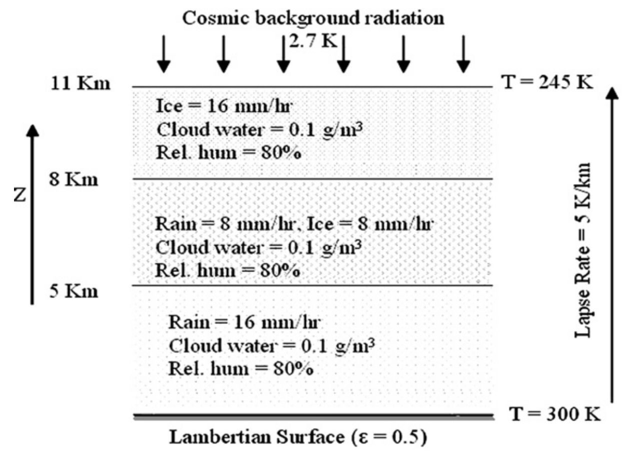

Fig. 4. Schematic of the three layer atmosphere.

Table 6
Comparison of albedo values for semi infinite atmospheres with those of Garcia and Siewert

ω_0	S	DDOM 8-stream			DDOM 16-stream			Garcia and Siewert [16]
		Number of layers			Number of layers			
		10	50	100	10	50	100	
0.7	1	0.15022	0.15511	0.15524	0.15020	0.15507	0.15520	0.15524
	10	0.23412	0.23539	0.23543	0.23410	0.23537	0.23541	0.23542
	100	0.25389	0.25406	0.25407	0.25387	0.25404	0.25404	0.25404
0.9	1	0.21626	0.22407	0.22429	0.21625	0.22403	0.22425	0.22431
	10	0.39204	0.39533	0.39542	0.39203	0.39532	0.39541	0.39545
	100	0.46385	0.46453	0.46455	0.46384	0.46452	0.46454	0.46454
1	1	0.25576	0.26557	0.26584	0.25576	0.26553	0.26580	0.26589
	10	0.52487	0.53103	0.53121	0.52486	0.53103	0.53121	0.53127
	100	0.73851	0.74194	0.74205	0.73852	0.74194	0.74205	0.74204

Table 7
Interaction parameters for each layer of the three layer atmosphere model

	Frequency (GHz)					
	6.6	10.7	18.0	37.0	85.6	183.0
0–5 km Layer						
$\sigma_{a\eta}$, km^{-1}	0.02112	0.09124	0.267072	0.542530	1.47147	13.9216
$\sigma_{s\eta}$, km^{-1}	0.00088	0.00676	0.053928	0.45747	1.25853	1.4784
g	0.091	−0.17	-0.082	0.010	0.276	0.539
5–8 km Layer						
$\sigma_{a\eta}$, km^{-1}	0.01171	0.035872	0.106875	0.380844	0.90372	7.0300
$\sigma_{s\eta}$, km^{-1}	0.000288	0.002128	0.018125	0.215156	1.13628	2.3970
g	0.045	0.014	−0.010	0.091	0.394	0.522
8–11 km Layer						
$\sigma_{a\eta}$, km^{-1}	0.00187	0.004998	0.014191	0.045318	0.1218	3.1624
$\sigma_{s\eta}$, km^{-1}	0.000128	0.001002	0.008809	0.136682	1.3282	2.7376
g	0.012	0.031	0.087	0.305	0.516	0.539

3.4. Atmospheric radiative transfer problem

To check the applicability of the present method for atmospheric radiative transfer problems, a simplified three layer atmosphere is considered [17]. The atmosphere is modeled as a three layer absorbing, emitting and scattering participating medium, as shown in Fig. 4. The bottom surface emissivity, temperature and lapse rate are assumed to be 0.5, 300 K and 5 K/km, respectively. The medium considered here is a non-gray medium. Hence, the radiative properties are also a function of wave length or frequency. A constant relative humidity of 80% is set throughout the cloud. The non-precipitating cloud liquid water is also assumed constant with a value of 0.1 g/m^3. The hydrometeor profile is assumed as 16 mm/h throughout the cloud. In the bottom most layer, the hydrometeors are assumed liquid. The hydrometeors are equally divided among liquid and frozen drops in the middle layer and in the top layer, all the hydrometeors are assumed frozen. The values of the absorption coefficient, scattering coefficient, asymmetry factor for each layer of the atmosphere for various microwave frequencies [17] are given in Table 7. The expansion coefficients for the anisotropic phase function are taken from [17].

The radiative transfer equation for the simplified three layered atmosphere is solved using the multilayer DDOM to obtain the radiances or intensities leaving the top of the atmosphere at a viewing angle of 50°.

As already discussed, with the multilayer DDOM giving accurate solutions for gray participating media, it is expected to do the same for multilayer non-gray atmospheres too. The brightness temperatures at different microwave frequencies obtained from the present study are compared with various models available in literature (Table 8). For this simple case, the differences in brightness temperature between present work and the other models are small at lower frequency channels. However, at higher frequency channels the maximum difference is about 4 K. This difference is more pronounced with an increase in optical thickness and complexity of scattering.

Table 8
Brightness temperatures by various methods for different microwave frequencies

Method	Frequency (GHz)					
	6.6	10.7	18.0	37.0	85.6	183.0
Eddington [17]	203.4	259.9	261.9	216.9	158.3	228.9
16 stream DOM [17]	203.6	260.4	262.4	217.1	159.4	230.0
Finite volume [18]	204.5	260.1	261.5	215.0	156.9	231.6
Doubling & Adding [19]	204.7	260.9	260.9	215.2	156.1	231.3
DDOM (Present)	203.4	260.7	264.2	218.4	159.3	232.3
DDOM (Present) with Planck	203.5	260.9	264.6	219.3	161.3	237.1

All the methods use the Rayleigh–Jeans approximation (RJ) to calculate the brightness temperatures. In the present work, both Planck's blackbody function and, Rayleigh–Jeans approximation were used. The brightness temperatures obtained from Planck's blackbody function from DDOM are also shown in Table 8. The error in the brightness temperature due to RJ approximation is about 5 K at 183 GHz. This shows that for higher frequency channels the use of Planck's blackbody function is a necessity.

4. Conclusions

A multilayer differential discrete ordinates method (DDOM) for radiative transfer through one-dimensional inhomogeneous participating media has been proposed and validated extensively with several results available in literature. Three validation cases for the multilayer model are presented to test its capability to handle inhomogeneous participating media. The results in all the three cases agree closely with those of benchmark cases (F_N method). This method contains no complicated mathematics and avoids tedious programming effort. It can easily incorporate variations in boundary conditions.

DDOM is then applied to the problem of atmosphere radiative transfer by considering a simple three layer atmospheric model, in which the medium is non-gray. The calculated brightness temperatures at various frequencies are on par with those obtained by several other numerical methods available in the literature. It is also shown that calculation of brightness temperature using the Rayleigh–Jeans approximation leads to an error of about 5 K at higher frequencies (at 183 GHz). Hence, the differential discrete ordinate method can be easily applied to practical radiative transfer problems to get fairly accurate results with less effort. The DDOM thus has tremendous application potential in remote sensing, where many international satellite missions even today use the highly simplified two stream analytical Eddington approach. Any method that improves the accuracy of radiative transfer models for the atmosphere has the potential to reduce the error in inverting satellite radiation intensities to vertical profiles of temperature, humidity, pressure or precipitation parameters.

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